A progress on the shifted power function for modeling informetric laws

Fred Y. Ye
School of Information Management,
Nanjing University, Nanjing 210093 CHINA
e-mail: yye@nju.edu.cn

ABSTRACT

Based on three theorems for size-to-rank transformations, the shifted power function is used to set up a theoretical framework unifying informetric distributions. While the shifted power function with time-type exponential cutoff supplies a unifying informetric framework, the shifted power function with space-type exponential cutoff can be linked to the Pareto and Weibull distributions. The exponent is the crucial parameter in a power function, determining the most important characteristics of the power-law distribution.

Keywords: Power law function; Exponential cutoff; Pareto distribution; Weibull distribution; Informetric functions.

INTRODUCTION

It is well known that many distributions in informetrics and information networks follow a power-law (some with an exponential cutoff) (Egghe 2005a; Newman 2010). While the shifted Lotka theory (Egghe and Rousseau 2012; Burrell 2008) was discussed recently, a shifted power function can be considered as a general unifying function for informetric distributions. Combining network studies (Albert and Barabási 2002; Newman 2003, 2005, 2010) with informetrics provides an interesting theoretical development, enriching informetric studies.

Following Egghe, (2005a; 2005b) as well as Egghe and Rousseau (2012), we first set up the theoretical foundation of the shifted power law function before studying further extensions.

THEORETICAL FOUNDATION: SIZA-RANK TRANSFORMATION

Let us focus on the general shifted power function f(x) changed by variable x

$$f(x) = c(x+d)^{-\alpha} \tag{1}$$

where *α>0* is power exponent and c and d are constants.

At first, we recall three theorems for size-rank transformation proofed by Egghe and Rousseau, with different forms by emphasizing the roles of different integral intervals.

Link Lotka size distribution to Zipf-Mandelbrot rank function

Theorem 1 (Egghe 2005a, 2005b): When *α>0* and α≠1, size variable x changes to rank variable r with interval transformation $x \in [1, N] \rightarrow r \in [1, T]$, which will result in size-rank transformation from the standard Lotka size distribution

$$f(x) = cx^{-\alpha} \tag{2}$$

to Zipf-Mandelbrot rank function

$$g(r) = p(r+q)^{-\beta} \tag{3}$$

where N denotes the maximal item per source density and T denotes the total number of sources, and c, p, q are constants.

Proof:

In Eq.(2), f(x) is a size-frequency distribution. Suppose its corresponding rank-frequency distribution function be g(r), following (Egghe 2005a; 2005b)

$$r = g^{-1}(x) = \int_x^N f(t)dt \tag{4}$$

$$f(x) = -\frac{1}{g'(g^{-1}(x))} \tag{5}$$

While x→r, interval transformation is $x \in [1, N] \rightarrow r \in [1, T]$. Put Eq.(2) into Eq.(4), we derive

$$r = \int_x^N ct^{-\alpha}dt = \frac{c}{1-\alpha}(N^{1-\alpha} - x^{1-\alpha}) \tag{6}$$

Then we solve

$$x = g(r) = \left(\frac{\alpha-1}{c}r + \frac{1}{N^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} = \left[\frac{cN^{\alpha-1}}{(\alpha-1)N^{\alpha-1}r + c}\right]^{\frac{1}{\alpha-1}} = \frac{\left(\frac{c}{\alpha-1}\right)^{\frac{1}{\alpha-1}} N^{2-\alpha}}{\left(r + \frac{cN^{1-\alpha}}{\alpha-1}\right)^{\frac{1}{\alpha-1}}} \tag{7}$$

Let

$$p = (\frac{c}{\alpha - 1})^{\frac{1}{\alpha-1}} N^{2-\alpha} \tag{8}$$

$$q = \frac{c}{\alpha - 1} N^{1-\alpha} \tag{9}$$

$$\beta = \frac{1}{\alpha - 1} \tag{10}$$

We obtain Zipf-Mandelbrot distribution Eq.(3) from Eq.(7).

Inversely, from Eq. (3), following Eq.(5), we have

$$f(x) = -\frac{1}{g'(g^{-1}(x))} = -\frac{1}{-\beta p[(\frac{p}{x})^{1/\beta} - q + q]^{-(\beta+1)}} = \frac{\frac{1}{\beta p} p^{\frac{\beta+1}{\beta}}}{x^{\frac{\beta+1}{\beta}}}$$

Let

$$c = \frac{1}{\beta p} p^{\frac{\beta+1}{\beta}} \tag{11}$$

$$\alpha = \frac{\beta + 1}{\beta} \tag{12}$$

We reduce to Eq.(2).

Corollary: Interval transformation $x \in [1, \infty[\ \rightarrow \ r \in]0, T]$ will result in size-rank transformation from standard Lotka size distribution Eq.(2) to standard Zipf rank function

$$g(r) = br^{-\beta} \tag{13}$$

Proof:

If the interval transformation is $x \in [1, \infty[\rightarrow r \in]0, T]$ while x→r, the result becomes

$$r = \int_x^\infty ct^{-\alpha} dt = \frac{c}{\alpha - 1} x^{1-\alpha} \tag{14}$$

then

$$x = g(r) = (\frac{\alpha - 1}{c} r)^{\frac{1}{1-\alpha}} = \frac{(\frac{c}{\alpha - 1})^{\frac{1}{\alpha-1}}}{r^{\frac{1}{\alpha-1}}} \tag{15}$$

Introducing

$$b = (\frac{c}{\alpha - 1})^{\beta} \tag{16}$$

and keeping $\beta=(\alpha-1)^{-1}$, we just obtain standard Zipf function, Eq. (13). With similar process by using Eq.(5), we can reduce to Eq.(2) from Eq.(13).

Link shifted Lotka distribution to shifted rank function

***Theorem 2*:** When *α>0* and α≠1, size variable x changes to rank variable r with interval transformation $x\in[1,N]\to r\in[1,T]$, which will result in size-rank transformation from shifted Lotka size distribution

$$f(x)=c(x+d)^{-\alpha} \tag{17}$$

to shifted rank function

$$g(r)=p(r+q)^{-\beta}-d \tag{18}$$

where N denotes the maximal item per source density and T denotes the total number of sources, and c, d, p, q are constants.

Proof:

Following similar process of shifted Lotka theory (Egghe and Rousseau 2012), by using same Eqs. (4) and (5), we derive results as follows.

While x→r, interval transformation is $x\in[1,N]\to r\in[1,T]$. We derive

$$r=\int_x^N c(t+d)^{-\alpha}dt=\frac{c}{1-\alpha}[(N+d)^{1-\alpha}-(x+d)^{1-\alpha}] \tag{19}$$

Then we solve

$$x=g(r)=(\frac{\alpha-1}{c}r+\frac{1}{(N+d)^{\alpha-1}})^{\frac{1}{1-\alpha}}-d=[\frac{c(N+d)^{\alpha-1}}{(\alpha-1)(N+d)^{\alpha-1}r+c}]^{\frac{1}{\alpha-1}}-d=\frac{(\frac{c}{\alpha-1})^{\frac{1}{\alpha-1}}(N+d)^{2-\alpha}}{[r+\frac{c(N+d)^{1-\alpha}}{\alpha-1}]^{\frac{1}{\alpha-1}}}-d \tag{20}$$

Let

$$p=(\frac{c}{\alpha-1})^{\frac{1}{\alpha-1}}(N+d)^{2-\alpha} \tag{21}$$

$$q=\frac{c}{\alpha-1}(N+d)^{1-\alpha} \tag{22}$$

$$\beta=\frac{1}{\alpha-1} \tag{23}$$

So we obtain shifted rank distribution Eq. (18) according to Eq. (20). Inversely, from Eq. (18), following Eq. (5), we have

$$f(x) = -\frac{1}{g'(g^{-1}(x))} = -\frac{1}{-\beta p[(\frac{p}{x+d})^{1/\beta} - q + q]^{-(\beta+1)}} = \frac{\frac{1}{\beta p} p^{\frac{\beta+1}{\beta}}}{(x+d)^{\frac{\beta+1}{\beta}}} \tag{24}$$

Let

$$c = \frac{1}{\beta p} p^{\frac{\beta+1}{\beta}} \tag{25}$$

$$\alpha = \frac{\beta+1}{\beta} \tag{26}$$

We reduce to Eq. (17).

Corollary: Interval transformation $x \in [1, \infty[\rightarrow r \in]0, T]$ will result in size-rank transformation from shifted Lotka size distribution Eq.(17) to the special shifted rank function

$$g(r) = br^{-\beta} - d \tag{27}$$

Proof:

If the interval transformation is $x \in [1, \infty[\rightarrow r \in]0, T]$ while x→r, the results become

$$r = \int_x^{\infty} c(t+d)^{-\alpha} dt = \frac{c}{\alpha - 1}(x+d)^{1-\alpha} \tag{28}$$

then

$$x = g(r) = (\frac{\alpha-1}{c} r)^{\frac{1}{1-\alpha}} - d = \frac{(\frac{c}{\alpha-1})^{\frac{1}{\alpha-1}}}{r^{\frac{1}{\alpha-1}}} - d \tag{29}$$

Introducing

$$b = (\frac{c}{\alpha-1})^{\beta} \tag{30}$$

and keeping $\beta = (\alpha-1)^{-1}$, we obtain the special shifted rank function Eq. (27). With similar process by using Eq.(5), we can reduce to Eq.(17) from Eq.(27).

When d=1, it is just the case of shifted Lotkaian function (Egghe and Rousseau 2012).

Link size-frequency power function and rank-frequency exponential function

Theorem 3: When α=1, size variable x changes to rank variable r with interval transformation $x \in [1, N] \to r \in [1, T]$, which will result in frequency power function

$$f(x) = c(x + d)^{-1} \tag{31}$$

to rank-frequency exponential function

$$g(r) = pe^{-qr} - d \tag{32}$$

where c>0, ∞>d≥0 , p>1, ∞>q>0 are constants.

Proof:

Following similar process of (Egghe and Rousseau 2003), by using Eqs. (4) and (5), we derive as follows.

While x→r, interval transformation is $x \in [1, N] \to r \in [1, T]$. We derive

$$r = \int_x^N c(t + d)^{-1} dt = c \ln \frac{N + d}{x + d} \tag{33}$$

Then we solve

$$x = g(r) = (N + d)e^{-\frac{r}{c}} - d \tag{34}$$

Let

$$p = N + d \tag{35}$$

$$q = \frac{1}{c} \tag{36}$$

So we obtain shifted rank distribution Eq. (32) according to Eq. (34).
Inversely, from Eq. (32), following Eq.(5), we have

$$f(x) = -\frac{1}{g'(g^{-1}(x))} = -\frac{1}{-pqe^{-q(-\frac{1}{q}\ln\frac{x+d}{p})}} = \frac{p}{q(x + d)} \tag{37}$$

Let

$$c = \frac{p}{q} \tag{38}$$

We reduce to Eq.(31).

When d=0, it is just the case proofed by Egghe and Rousseau (2003), which shows that a size-frequency power function with α=1 is equivalent with an exponentially decreasing rank-frequency function.

THORETICAL EXTENSION I: SHIFTED POWER FUNCTION WITH TIME-TYPE EXPONENTIAL CUTOFF

Following classic informetric models (Leimkuhler 1967; Fairthome 1969; Price 1976; Garfield 1980; Mandelbrot 1982; Bookstein 1990; Egghe and Rousseau 1995), a unified informetric model via a simple distribution function (Ye and Rousseau 2010; Ye 2011) has been proposed.

$$f(x,t) = c(x+d)^{-\alpha} e^{kt} \tag{39}$$

where t denotes time and x denotes spatial (size-frequency) variable.

Now, some supplements and corrections on this unified informetric model as given. In following sections, f(x) always denotes various size-frequency distributions and g(r) various rank-frequency distributions, while F(x) and G(r) denotes the corresponding cumulative distributions of f(x) and g(r) respectively. While f(x) means space-type distribution, f(t) does time-type distribution.

When x→r and interval transformation as $x \in [1, N] \to r \in [1, T]$ or $r \in [0, T]$, based on Eq. (39), we suppose similar result according to above Theorem 2.

$$g(r,t) = p(r+q)^{-\beta} e^{\lambda t} - d \tag{40}$$

where a constant difference (-d) may happen at most. Because the exponential item can be accounted into constant p (there is no changes about t), the time-type exponential cutoff never affect the size-rank transformation.

As size function Eq. (39) satisfied wave-heat equations (Ye 2011) and the differential of a constant is zero, rank function Eq. (40) will also satisfies the wave-heat equations. This is why we have similar mathematical structure for both size variable x and rank variable r (Ye and Rousseau 2010).

Space-type distributions

Generally, we can consider Eqs. (1) and (3) are basic spatial functions for size- frequency distribution and rank-frequency distribution respectively. When we ignore time factor (exponential item), cumulative rank distributions G(r) can be derived with integral on interval $r \in [1, r]$ or $r \in]0, r]$ (there is no ∞ for rank) when β≠1 as

$$G(r) = \int_1^r p(r+q)^{-\beta} dr = \frac{p}{1-\beta}[(r+q)^{1-\beta} - (1+q)^{1-\beta}] \tag{41}$$

$$G(r) = \int_0^r p(r+q)^{-\beta} dr = \frac{p}{1-\beta}[(r+q)^{1-\beta} - q^{1-\beta}] \tag{42}$$

These are cumulative Zipf-Mandelbrot-type distributions, which also belong to generalized Leimkuhler functions (Rousseau 1988).

When β=1, the integral produce regular Leimkuhler-type distributions

$$G(r)=\int_1^r p(r+q)^{-1}dr = p\ln[(r+q)/(1+q)] \tag{43}$$

$$G(r)=\int_0^r p(r+q)^{-1}dr = p\ln(1+r/q) \tag{44}$$

In Eq.(3), when β=1and q=0, it is just Zipf's law.

So, the space-type distributions from the unified informetric functions Eqs. (39) and (40) cover all forms of Bradford, Lotka, Zipf, Leimkuhler, and Mandelbrot laws as shown in Table 1.

Table 1: Typical Space-type Distribution as Informetric Laws

x or r	α or β	d or q	Informetric law	Mathematical form
x: size approach	α≠1	d≠0	Bradford-Vickery type	$f(x)=c(x+d)^{-\alpha}$
	α=1	d≠0	Bradford-Brookes type	$F(x)=c\ln[(r+d)/(1+d)]$ $F(x)=c\ln(1+r/d)$
	α~2	d=0	Lotka	$f(x)=cx^{-\alpha}$
r: rank approach	β≠1	q≠0	Zipf-Mandelbrot; Generalized Leimkuhler	$g(r)=p(r+q)^{-\beta}$; $G(r)=\frac{p}{1-\beta}[(r+q)^{1-\beta}-(1+q)^{1-\beta}]$ $G(r)=\frac{p}{1-\beta}[(r+q)^{1-\beta}-q^{1-\beta}]$
	β=1	q≠0	Leimkuhler-type	$G(r)=p\ln[(r+q)/(1+q)]$ $G(r)=p\ln(1+r/q)$
	β=1	q=0	Zipf	$g(r)=pr^{-1}$

Since the space-type distribution of the unified informetric model covers all forms of Bradford, Lotka, Zipf, Leimkuhler and Mandelbrot distribution (Fairthome 1969; Bookstein 1990; Rousseau 1990), we can say that it reveals a rich and colorful unified mechanism.

Time-type distributions

When we only consider time distribution, the distribution is independence of space so that we have (Ye 2011).

$$\frac{d^2 f}{dt^2} - \frac{c^2}{d^2}\frac{df}{dt} = c^2 (b-a) f(t) \tag{45}$$

where a, b, c and d are real constants.

The characteristic equation of Eq.(45) is $r^2 - (c^2/d^2)r - c^2(b-a) = 0$. Its two solutions are $r_{1,2} = [(c^2/d^2) \pm \sqrt{(c^4/d^4) + 4c^2(b-a)}]/2$ so that the general solution of Eq. (45) becomes

$$f(t) = c_1 e^{r_1 t} + c_2 e^{-r_2 t} \tag{46}$$

In the solutions of (46), it covers some special time-type distributions of informetrics, as shown in Table 2.

Table 2: Some Special Time-type Distributions of Informetrics

r_1, r_2	c_1, c_2	Informetric type	Mathematical form
$r_1, r_2>0$	$c_1, c_2 \neq 0$	Avramescu-type	$f(t) = c_1 e^{r_1 t} + c_2 e^{-r_2 t}$
$r_1>0$ $r_2=0$	$c_2 \neq 0$	Exponential-type	$f(t) = c_1 e^{r_1 t} + c_2$
	$c_2=0$	Price-type (growth)	$f(t) = c_1 e^{r_1 t}$
$r_1=0$ $r_2>0$	$c_1 \neq 0$	Negative Exponentail-type	$f(t) = c_1 + c_2 e^{-r_2 t}$
	$c_1=0$	Bernal-Brookes-type (ageing)	$f(t) = c_2 e^{-r_2 t}$

Combining (46) with Table 2, we see that most known time-type informetric distributions are included.

Another possible or potential study concerns whether integrated transformation can link space-type distributions with time-type ones.

Clearly the function $f(x,t) = c(x+d)^{-\alpha} e^{kt}$ or $g(r,t) = p(r+q)^{-\beta} e^{\lambda t}$ is a simple and interesting common solution of the partial differential equations (Ye and Rousseau 2010; Ye 2011), belonging to wave-heat equations. The unified informetric model avoids some artificial suppositions such as another unified scientometric model (Bailón-Moreno et al. 2005). Meanwhile, both space-type and time-type distributions may introduce new types beyond the present informetric laws, which provide a theoretical framework and could stimulate further studies.

THEORETICAL EXTENSION II: SHIFTED POWER FUNCTION WITH SPACE-TYPE EXPONENTIAL CUTOFF

When we consider the shifted power function with exponential spatial cutoff

$$f(x) = c(x+d)^{-\alpha} e^{-\kappa x} \tag{47}$$

it provides a theoretical path to link with other known distributions.

As a probability density function (PDF) f(x), Eq.(47) provides the way to approach other probability distributions (reference system see Appendix).

There are one constant c and three independent parameters as α, κ and d in Eq. (47). If κ=0 and d=0, it becomes standard power function as Pareto distribution (Type I).

When α=0, the PDF (47) becomes a pure exponential distribution and the constant c is simple according to normalization condition for $x \geq x_m$

$$c = \kappa e^{\kappa x_m} \tag{48}$$

where x_m is the minimum x.

When κ=0, the PDF (47) becomes shifted power function (1) and the normalized constant c is

$$c = (\alpha - 1)(x_m + d)^{\alpha - 1} \tag{49}$$

So the shifted power law density distribution (47) becomes

$$f(x) = (\alpha - 1)(x_m + d)^{\alpha - 1}(x+d)^{-\alpha} \tag{50}$$

with normalized condition

$$\int_{x_m}^{\infty} f(x)dx = \int_{x_m}^{\infty} c(x+d)^{-\alpha} e^{-\kappa x} dx = 1 \tag{51}$$

Corresponding cumulative distribution function (CDF) is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} (\alpha - 1)(x_m + d)^{\alpha - 1}(t+d)^{-\alpha} dt \tag{52}$$

If x belongs to discrete variables, as $c\sum_{\alpha=1}^{\infty} x^{-\alpha} = c\zeta(\alpha) = 1$, we have

$$c = 1/\zeta(\alpha) \tag{53}$$

where ζ(α) is Riemann ζ-function.

In very special case, when we set x → x/λ, β=1-k, κx=$(x/\lambda)^k$ and c=k/λ, the PDF (47) becomes Weibull distribution

$$f_{k,\lambda}(x)=\frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}e^{-(\frac{x}{\lambda})^k} \tag{54}$$

where shape parameter k>0 and scale parameter λ>0 and x≥0 (When x<0, f(x)=0).

If κ≠0, generally, according to normalized condition, the constant c in Eq.(47) can be determined by integral

$$\int_{x_m}^{\infty} f(x)dx=\int_{x_m}^{\infty} c(x+d)^{-\alpha}e^{-\kappa x}=1 \tag{55}$$

For λ>0, integral formula is

$$\int_0^{\infty} x^{n-1}e^{-\kappa x}dx=\frac{1}{\kappa^n}\Gamma(n) \tag{56}$$

So we have (Clauset et al., 2009)

$$c=\frac{\kappa^{1-\alpha}}{\Gamma(1-\alpha,\kappa x_m)} \tag{57}$$

in which Γ denotes Γ-function. According to Stirling's approximation for the gamma function, it satisfies

$$\Gamma(x)\approx\sqrt{2\pi}e^{-x}x^{x-1/2} \tag{58}$$

DISCUSSION

Obviously, all power functions or Pareto distributions are crucially determined by exponent α, so that the exponent α is the most important measure in power-law distributions. Statistically, its moments are given by (for k < α -1).

$$<x^k>=\int_{x_{mi}}^{\infty} x^k f(x)dx=\frac{\alpha-1}{\alpha-1-m}x_m^k \tag{59}$$

which is strongly linked with power exponent α. When 1 < α < 2, the first moment, i.e. the mean, is infinite, along with all higher moments. When 2 < α < 3, the first moment is finite, but the second one (the variance) and higher moments are infinite. That is, all moments k ≥ α -1 diverge: when α < 2, the average and all higher-order moments are infinite; when 2 < α < 3, the mean exists, but the variance and higher-order moments are infinite. For finite-size samples drawn from such distribution, this behavior implies that the central moment estimators (like the mean and the variance) for diverging moments will never converge.

Meanwhile, the Gini coefficient, which is another measure of the deviation of the Lorenz curve, linking Pareto distribution, from the equidistribution line which is a line connecting [0, 0] and [1, 1], is also determined by α. The Gini coefficient for the Pareto distribution is calculated according to following formula.

$$G = 1 - 2\int_0^1 L(F)dF = \frac{1}{2\alpha - 1} \tag{60}$$

Generally, two cases are included, i.e. the convex form as Lorenz curve ranking from smallest to largest and concave form as Leimkuhler curve ranking from largest to smallest (Burrell 2005, 2007).

The Lorenz curve is often used to characterize income and wealth distributions. For any distribution, when the Pareto index is α = $\log_4(5)$=log(5)/log(4), approximately 1.161, 80/20 rule can be derived from the Lorenz curve formula given above. This excludes Pareto distributions in which $0 < \alpha \leq 1$, which have infinite expected value, and so cannot reasonably model income distribution.

To find the estimator for α, we can compute the corresponding partial derivative and determine where it is zero

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} + n\ln x_m - \sum_{i=1}^{n} \ln x_i = 0 \tag{61}$$

Thus following formula becomes the maximum likelihood estimator for α, with expected statistical error $\sigma = \alpha_{\max} / \sqrt{n}$

$$\alpha_{\max} = n / \sum_{i=1}^{n} (\ln x_i - \ln x_m^{\max}) \tag{62}$$

The power exponent also links with fractal dimension D as a self-similar fractal as D = α–1 (Egghe 2009). So we can say, in most general senses, the power exponent α is so important that it determines main properties of power-law distributions.

On the other hand, supposing the power function g(r) distribute from high to low according to the rank variable r, referring to Hirsch (2005), the intersection of the 45° line with the curves will give a series of h-index, as shown in Figure 1.

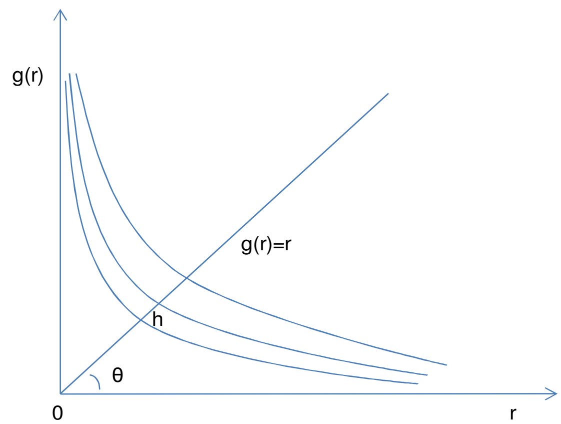

Figure1: h-index in Power-law Distribution

In Figure 1, h-points are produced by the crosses of power function g(r) ~ $r^{-\beta}$ and the 45° line g(r)=(tanθ)r, where θ=45° and tan45°=1. Statistically, h-index is a characteristic indicator for differentiating the curvatures of different power functions, while α marks the shape of a power function.

In addition, we should mention the limitations of above processing. As shifted power function was set to an ideal-like or simple model, it might differentiate from real situations, so that there could be differences between the ideal model and real states. Moreover, as an ideal or standard model is benefit for theoretical analysis, the simple approach is useful for further studies.

CONCLUSION

The general shifted power function leads to a unified core and provides a unified and rich mechanism of informetric distributions. Based on the size-to-rank transformation, most theoretical issues can be linked together. While the shifted power function with exponential time cutoff supplies a unified informetric function, the shifted power function with exponential spatial cutoff can – under special situations – be linked to the Pareto and Weibull distributions. The power exponent and h-index are two crucial parameters of the power functions, which determine the most important characteristics of the power law distributions. If the general shifted power function becomes the core of standard model for informetrics, more developments can be expected.

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