On Power Generalized Akash Distribution with Properties and Applications

Samuel Aderoju[1*] & **Isaac Adeniyi**[2]

[*1]*Department of Statistics and Mathematical Sciences, Kwara State University, Malete, P.M.B. 1530, Ilorin, Kwara State, Nigeria.*

[2]*Department of Mathematical Sciences, Federal University Lokoja, P.M.B. 1154, Lokoja, Kogi State, Nigeria.*

**Corresponding author:* [*1]samuel.aderoju@kwasu.edu.ng

Abstract

A new lifetime model known as power generalized Akash distribution (PGAD), which extends generalized Akash (GA) distribution has been proposed in this paper. The PGAD was inspired by the wide use of the Akash and GA distributions in various applied areas. Some structural characteristics of the new model were studied such as moments, reliability, hazard rate function, survival function, Renyi entropy measure and order statistics. The parameters of the model were obtained via the maximum likelihood estimation method. The flexibility and importance of the new distribution has been illustrated by its applications to two real datasets. Using BIC, AIC and -2Loglikelihood, it is obvious that the PGAD is more effective than Topp-Leone Lomax (TLLo), Generalized Akash (GA), Power Pranav (PP) and Power Transformed Power Inverse Lindley (APTPIL) distributions in modelling real lifetime data.

Keywords: power transformation, Akash distribution, reliability, order statistics, Maximum likelihood estimator

1. Introduction

Recently, different lifetime distributions have been proposed for modeling survival (or "time-to-event") data since the classical one parameter Lindley distribution (Lindley, 1958) lacks the flexibility required to model lifetime data exhibiting different shapes, such as, increasing, decreasing, bathtub, and a broad variety of monotone failure rates. Some of the recent lifetime distributions proposed by different researchers are Lomax-Cauchy distribution (Amalare et al., 2020). Shanker and Shukla (2020) proposed a new Quasi Sujatha distribution while Tesfay and Shanker (2019) introduced a generalized Sujatha distribution and New Generalized Poisson-Sujatha distribution proposed by Aderoju (2020). The authors studied the properties and applications of the distributions.

Many researchers have developed different power transformation and other form of generalization of Lindley, exponential, Weibull and other distributions. Mahdavi and Kundu (2016) introduced an extra parameter to a family of distributions for more flexibility. A special case was considered in details namely; one parameter exponential distribution. Various properties of the proposed distribution, including explicit expressions for the moments were derived. They also considered an extension of the two-parameter exponential distribution, mainly for data analysis purposes. This proposed distribution has several desirable properties, and they are quite similar to the corresponding properties of the well-known Gamma or Weibull family. One data analysis has been performed, and it is observed that the proposed model provides a better fit than some of the existing models.

Asiribo et al. (2019) presented a four-parameter distribution known as the Lomax-Kumaraswamy distribution. The authors studied some properties of the model. The distribution is a positively skewed. "The implications of the plots for the survival function indicate that the Lomax-Kumaraswamy distribution could be used to model time or age-dependent events, where survival rate decreases with time or age. The performance of the Lomax-Kumaraswamy distribution was tested by using to two real datasets in the literature. The results showed that it can serve as an alternative distribution to model positively skewed datasets", Asiribo et al. (2019).

Yousof et al. (2019) proposed a new two parameter lifetime model called the Xgamma Weibull (XGW) distribution. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments generating function were derived. The authors discussed the method of maximum likelihood for estimating model parameters. An application has been shown to illustrate that their proposed model provides consistently better fit than the other competitive models.

Umar and Zakari (2020) proposed Beta Odd Generalized Exponential (BOGE) distribution. Some mathematical properties of the model are derived. They presented and studied three special cases of the BOGE family of distribution. The authors studied the performance of the BOGE distribution through its application to two real datasets (Glass Fibres data and Precipitation data) and the results shown that it is better than some existing distributions.

A two-parameter generalized Akash distribution was originally proposed by Shanker et al. (2018) from two-component mixture of gamma $(3, \theta)$ and exponential (θ) distributions as follows:

$$f_{ga}(y|\alpha, \theta) = p f_e(y|\theta) + (1-p) f_g(y|3, \theta),$$

where the mixing proportion (p) is $p = \frac{\theta^2}{\theta+\alpha}$.

$$f_e(y|\theta) = \theta e^{-\theta y}$$

and

$$f_g(y|3, \theta) = \theta^3 y e^{-\theta y}$$

and the corresponding probability density function (pdf) of the generalized Akash distribution (GAD) was derived as

$$f_{ga}(y|\alpha,\theta) = \begin{cases} \frac{\theta^3}{2\alpha+\theta^2}(\alpha y^2+1)e^{-\theta y}, & for\ y,\alpha,\theta > 0 \\ 0, & elsewhere \end{cases} \tag{1}$$

The density has a scale parameter (α) and a shape parameter (θ). Furthermore, the cumulative density function (cdf) of (1) was given as:

$$F_{ga}(y) = 1 - \left[1 + \frac{\alpha\theta y(2+\theta y)}{\theta^2+2\alpha}\right]e^{-\theta y},\ \ y,\alpha,\theta > 0 \tag{2}$$

The goal of this paper is to derive a power generalized Akash distribution in order to get better flexibility compared to the other popular generalizations of Akash model. It would eventually be obvious that the proposed model contains certain sub-models such as Akash distribution, GAD and exponential distribution.

Hence, we discussed the proposed model in section two while some of the mathematical properties are presented in section three. The MLE of the model's parameters is discussed in section four and section five contains the application of the model to real lifetime data. Final conclusion is presented in section six.

2. Materials and Methods

We derive pdf of the power generalized Akash distribution as follows:

A random variable X is said to follow a power generalized Akash distribution (PGAD) if:

$$X|\alpha,\theta,\omega \sim PGA(\alpha,\theta,\omega)$$

Such that $f_{pga}(x)$ is the *pdf* of PGAD and

$$\int_0^\infty f_{pga}(x)dx = 1, \qquad for\ \ \alpha,\theta,\omega > 0.$$

Hence, the PGAD is represented as $PGA(\alpha,\theta,\omega)$.

Theorem 1: Power Transformation Method (Box & Cox, 1964); let Y be a non-negative variable with probability density function, $f_y(y)$ and cumulative density function, $F_y(y)$; with the power transformation $X = Y^{\omega^{-1}}$ the new corresponding probability density function (pdf) and cumulative density function (cdf) respectively become:

$$f_X(x) = \omega x^{\omega-1} f_X(y^\omega) \tag{3}$$

$$F_X(x) = F_X(y^\omega) \tag{4}$$

Proof: Suppose $Y|\alpha,\theta \sim GA(\alpha,\theta)$, then the pdf of random variable Y is given in (1). Substitute (1) and (2) into (3) and (4) respectively to obtain (5) and (6) as follows:

$$f_{pga}(x|\alpha,\theta,\omega) = \begin{cases} \frac{\omega\theta^3 x^{\omega-1}}{2\alpha+\theta^2}(\alpha x^{2\omega}+1)e^{-\theta x^\omega}, & for\ x,\alpha,\theta,\omega > 0 \\ 0, & elsewhere \end{cases} \tag{5}$$

The density (5) has a scale parameter (α) and a shape parameters $(\theta\ and\ \omega)$

$$F_{pga}(x) = 1 - \left[1 + \frac{\alpha\theta x^{\omega}(2 + \theta x^{\omega})}{\theta^2 + 2\alpha}\right] e^{-\theta x^{\omega}}, \quad x, \alpha, \theta, \omega > 0 \tag{6}$$

The (5) and (6) are the pdf and cdf of PGAD respectively. It can be shown clearly from (5) that

$$\int_0^{\infty} f_{pga}(x)\, dx = 1,$$

this shows that $f_{pga}(x)$ is a true pdf.

Special cases:

When $\omega = 1$ *and* $\alpha, \theta > 0$; the pdf in (5) reduces to the base model (GAD).

When $\alpha, \omega = 1$ *and* $\theta > 0$; it reduces to Akash distribution (Shanker, 2015).

When $\omega = 1, \alpha = 0$ *and* $\theta > 0$; it reduces to the exponential distribution.

The graphical representation of the shape of pdf and cdf of the proposed PGAD at varying values of the parameters are presented in Figures 1 and 2 respectively.

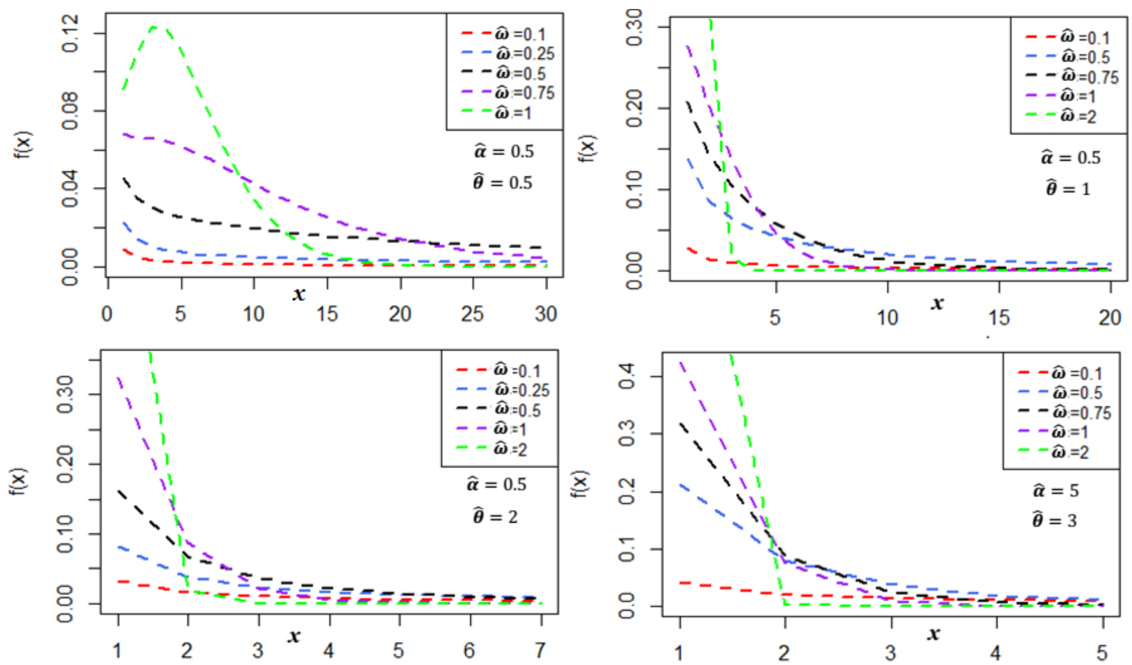

Figure 1: Plots of the pdf of the PGAD for different values of $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\omega}$.

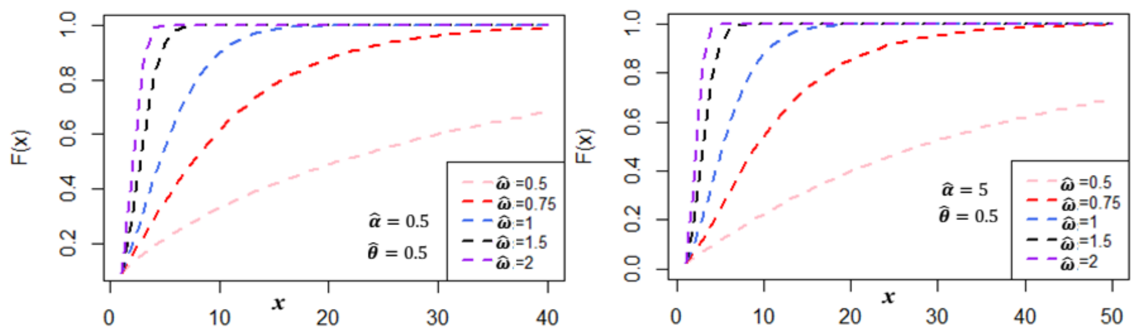

Figure 2: Plots of the *cdf* of the PGAD for different values of $\hat{\alpha}, \hat{\theta}$ *and* $\hat{\omega}$.

3. Mathematical Properties of Power Generalized Akash distribution

We present the moments, reliability analysis, entropy and the order statistics of the Power Generalized Akash distribution in this section.

3.1. Moments

Theorem 2: Suppose a random variable X follows power Generalized Akash distribution, that is $X \sim PGA(\alpha, \theta, \omega)$, then, the r^{th} order moment about origin is given by:

$$E(X^r) = \mu_r = \int_0^\infty x^r f(x) dx$$

Proofs: Hence, the first four moments are obtained as:

$$E(X^r) = \mu_r = \int_0^\infty x^r \frac{\omega\theta^3 x^{\omega-1}}{2\alpha + \theta^2}(\alpha x^{2\omega} + 1)e^{-\theta x^\omega} dx \qquad (7)$$

$$\mu_1 = \frac{\left(\alpha\omega\Gamma\left(3 + \frac{1}{\omega}\right) + \theta^2\Gamma\left(\frac{1}{\omega}\right)\right)}{(2\alpha + \theta^2)\theta^{1/\omega}}$$

$$\mu_2 = \frac{\left(\alpha\Gamma\left(3 + \frac{2}{\omega}\right) + \theta^2\Gamma\left(\frac{2+\omega}{\omega}\right)\right)}{(2\alpha + \theta^2)\theta^{2/\omega}}$$

$$\mu_3 = \frac{\left(\alpha\Gamma\left(3 + \frac{3}{\omega}\right) + \theta^2\Gamma\left(\frac{3+\omega}{\omega}\right)\right)}{(2\alpha + \theta^2)\theta^{3/\omega}}$$

$$\mu_4 = \frac{\left(\alpha\Gamma\left(3 + \frac{4}{\omega}\right) + \theta^2\Gamma\left(\frac{4+\omega}{\omega}\right)\right)}{(2\alpha + \theta^2)\theta^{4/\omega}}$$

Note that, the variance (σ^2) of the random variable X can be obtained as:

$$\sigma^2 = E(X^2) - [E(X^1)]^2 = \mu_2 - [\mu_1]^2$$

$$\therefore\ \sigma^2 = \frac{\left[(2\alpha+\theta^2)\left(\alpha\Gamma\left(3+\frac{2}{\omega}\right)+\theta^2\Gamma\left(\frac{2+\omega}{\omega}\right)\right) - \frac{\left(\alpha\omega\Gamma\left(3+\frac{1}{\omega}\right)+\theta^2\Gamma\left(\frac{1}{\omega}\right)\right)^2}{\omega^2}\right]}{\theta^{2/\omega}(2\alpha+\theta^2)^2}$$

The corresponding coefficient of variation (CV) of PGAD is obtained as:

$$CV = \frac{\sigma}{\mu_1}$$

$$= \frac{\sqrt{\omega\theta^{-1/\omega}\left[(2\alpha+\theta^2)\left(\alpha\Gamma\left(3+\frac{2}{\omega}\right)+\theta^2\Gamma\left(\frac{2+\omega}{\omega}\right)\right) - \frac{\left(\alpha\omega\Gamma\left(3+\frac{1}{\omega}\right)+\theta^2\Gamma\left(\frac{1}{\omega}\right)\right)^2}{\omega^2}\right]}}{(2\alpha+\theta^2)\left(\alpha\omega\Gamma\left(3+\frac{1}{\omega}\right)+\theta^2\Gamma\left(\frac{1}{\omega}\right)\right)}$$

3.2. Reliability Properties

In this section, the main reliability characteristics are derived in terms of the survival function, $S(x)$, as well as the hazard function, $h(x)$, of the PGAD.

3.2.1. Survival Function

The survival function is generally defined as the probability that an item does not fail prior to some time. It is expressed as:

$$S(x) = 1 - F(x) = 1 - \left\{1 - \left[1 + \frac{\alpha\theta x^\omega(2+\theta x^\omega)}{\theta^2+2\alpha}\right]e^{-\theta x^\omega}\right\}$$

$$S(x) = \left[1 + \frac{\alpha x^\omega\theta(2+\theta x^\omega)}{2\alpha+\theta^2}\right]e^{-\theta x^\omega}$$

3.2.2. Hazard rate function

The hazard rate function can be expressed as the conditional probability of failure, given that it has survived to the time. It is given as:

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\omega\theta^3 x^{\omega-1}(1+x^{2\omega}\alpha)}{\theta^2+\alpha(2+2x^\omega\theta+x^{2\omega}\theta^2)}$$

Figures 3 and 4 represent the graph of the survival function and hazard rate function of the Power Generalized Akash distribution, respectively, for varying values of the parameters α, θ and ω.

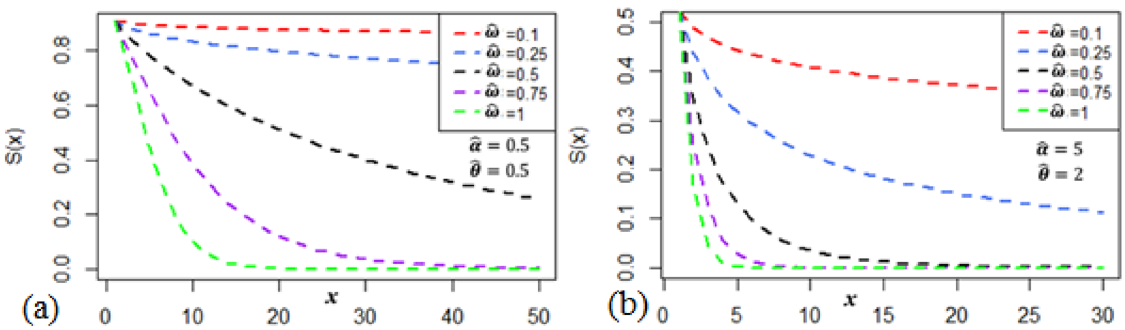

Figure 3: Survival function of the PGAD at different values of the parameters.

The graph of $S(x)$ of the $PGA(\alpha,\theta,\omega)$ for different values of the parameters (α,θ,ω) in Figure 3 shows that, the shapes of $S(x)$ is decreasing when $\theta \to 0$ and $\omega \to 0$ while it is constant when $\theta \to \infty$ and $\omega \to 1$; this also shows the flexibility of PGAD.

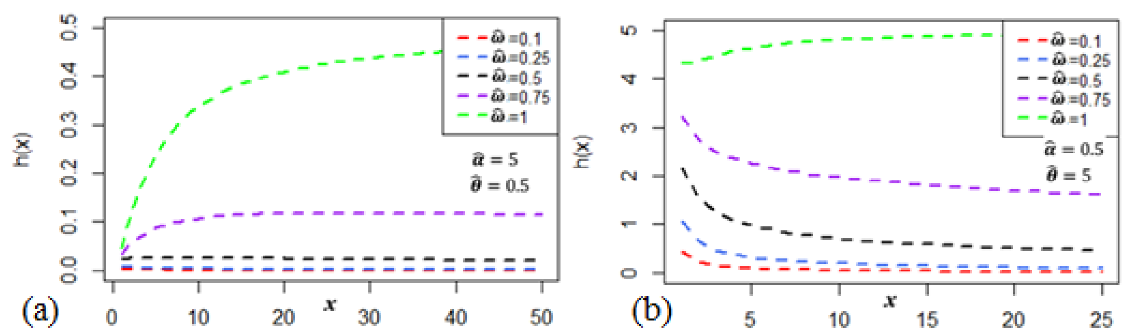

Figure 4: Hazard rate function of the PGAD at different values of the parameters.

The graphs of the hazard rate function of the PGAD for different values of the parameters are given in Figure 4. The model exhibits both monotone increasing and decreasing failure rate characteristic. It decreases monotonically when $\omega \to 0$ and increases monotonically when $\omega \to 1$.

3.3 Renyi's Entropy

An entropy of a random variable X is a measure of variation of uncertainty. One of the popular entropy measure is Renyi's entropy (Rényi, 1961). Suppose a continuous random variable X follows the Power Generalized Akash probability with density function $f_{pga}(x)$, then Renyi's entropy, $R_H(x)$, is defined as:

$$R_H(x) = \frac{1}{1-\gamma} log \int_0^\infty [f_{pga}(x)]^\gamma dx \tag{8}$$

$$= \frac{1}{1-\gamma} log \int_0^\infty \left[\frac{\omega\theta^3 x^{\omega-1}}{2\alpha+\theta^2}(\alpha x^{2\omega}+1)e^{-\theta x^\omega}\right]^\gamma dx$$

$$= \frac{1}{1-\gamma}\log\left[\frac{\omega^{\gamma}\theta^{3\gamma}}{(2\alpha+\theta^2)^{\gamma}}\int_0^{\infty} x^{(\omega-1)\gamma}(\alpha x^{2\omega}+1)^{\gamma}e^{-\theta\gamma x^{\omega}}dx\right]$$

$Let\ \ y = x^{\omega} \ \Rightarrow \ x = y^{\frac{1}{\omega}} \quad dx = \frac{1}{\omega y^{\omega-1}}dy$

$$= \frac{1}{1-\gamma}\log\left[\frac{\omega^{\gamma}\theta^{3\gamma}}{(2\alpha+\theta^2)^{\gamma}}\int_0^{\infty} \frac{1}{\omega}y^{(\gamma-{}^{\gamma}/_{\omega}+{}^{1}/_{\omega})-1}(\alpha y^2+1)^{\gamma}e^{-\theta\gamma x^{\omega}}dy\right]$$

$$= \frac{1}{1-\gamma}\log\left[\frac{\omega^{\gamma}\theta^{3\gamma}}{(2\alpha+\theta^2)^{\gamma}}\int_0^{\infty} \frac{y^{(\gamma-{}^{\gamma}/_{\omega}+{}^{1}/_{\omega})-1}}{\omega}\sum_{j=0}^{\infty}\binom{\gamma}{j}(\alpha y^2)^{j}e^{-\theta\gamma y}dy\right]$$

$$= \frac{1}{1-\gamma}\log\left[\frac{\omega^{\gamma}\theta^{3\gamma}}{(2\alpha+\theta^2)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{j}}{\omega}\int_0^{\infty} y^{(2j+\gamma-{}^{\gamma}/_{\omega}+{}^{1}/_{\omega})-1}\ e^{-\theta\gamma y}dy\right]$$

$$= \frac{1}{1-\gamma}\log\left[\frac{\omega^{\gamma}\theta^{3\gamma}}{(2\alpha+\theta^2)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{j}}{\omega}\frac{\Gamma\left(2j+\gamma-{}^{\gamma}/_{\omega}+{}^{1}/_{\omega}\right)}{(\theta\gamma)^{(2j+\gamma-{}^{\gamma}/_{\omega}+{}^{1}/_{\omega})}}\right]$$

3.4 Order Statistics

Suppose $X_1,\ X_2,\ \ldots, X_n$ is a random sample from PGA distribution, by definition, the k^{th} order statistic, $X_{(k)}$ can be expressed as:

$$f_{X(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!}f_{pga}(x)\left[F_{pga}(x)\right]^{k-1}\left[1-F_{pga}(x)\right]^{n-k} \tag{9}$$

Substituting (5) and (6) into (8), we have

$$f_{X(k)}(x) \ = \frac{n!\,\omega\theta^3x^{\omega-1}(\alpha x^{2\omega}+1)e^{-\theta x^{\omega}}}{(2\alpha+\theta^2)(k-1)!\,(n-k)!}\left[1-\left[1+\frac{\alpha\theta x^{\omega}(2+\theta x^{\omega})}{\theta^2+2\alpha}\right]e^{-\theta x^{\omega}}\right]^{k-1}$$

$$\times\left[\left[\frac{e^{-x^{\omega}\theta}(\theta^2+\alpha(2+2x^{\omega}\theta+x^{2\omega}\theta^2))}{2\alpha+\theta^2}\right]e^{-\theta x^{\omega}}\right]^{n-k}$$

The first and n^{th} orders are:

$$f_{X(1)}(x) = \frac{n\omega\theta^3x^{\omega-1}(\alpha x^{2\omega}+1)e^{-\theta x^{\omega}}}{(2\alpha+\theta^2)}\left[\left[\frac{e^{-x^{\omega}\theta}(\theta^2+\alpha(2+2x^{\omega}\theta+x^{2\omega}\theta^2))}{2\alpha+\theta^2}\right]e^{-\theta x^{\omega}}\right]^{n-1}$$

$$f_{X(n)}(x) = \frac{n\omega\theta^3x^{\omega-1}(\alpha x^{2\omega}+1)e^{-\theta x^{\omega}}}{(2\alpha+\theta^2)}\left[1-\left[1+\frac{\alpha\theta x^{\omega}(2+\theta x^{\omega})}{\theta^2+2\alpha}\right]e^{-\theta x^{\omega}}\right]^{n-1}$$

4. Maximum Likelihood Estimation

Suppose $X_1,,\ldots,X_n$ is a random sample of size n from the PGAD, the maximum likelihood function of

PGAD can be expressed as:

$$L(\alpha,\theta,\omega)=\prod_{i=1}^{n}\frac{\omega\theta^3 x_i^{\omega-1}}{2\alpha+\theta^2}\left(\alpha x_i^{2\omega}+1\right)e^{-\theta x_i^{\omega}},$$

hence, the log-likelihood is:

$$lnL=nln(\omega)+3nln(\theta)+(\omega-1)\sum_{i=1}^{n}lnx_i-nln(2\alpha+\theta^2)+\sum_{i=1}^{n}\ln\left(\alpha x_i^{2\omega}+1\right)-\theta\sum_{i=1}^{n}x_i^{\omega}$$

Differentiating the lnL partially with respect to associated parameters we have

$$\frac{\partial lnL}{\partial\alpha}=-\frac{2n}{2\alpha+\theta^2}+\sum_{i=1}^{n}x_i^{2\omega}\left(\alpha x_i^{2\omega}+1\right)^{-1}=0 \quad (10)$$

$$\frac{\partial lnL}{\partial\theta}=\frac{3n}{\theta}-\frac{2n\theta}{2\alpha+\theta^2}-\sum_{i=1}^{n}x_i^{\omega}=0 \quad (11)$$

$$\frac{\partial lnL}{\partial\omega}=\frac{n}{\omega}+\sum_{i=1}^{n}\ln(x_i)+\sum_{i=1}^{n}\frac{2\alpha x_i^{2\omega}\ln(x_i)}{\alpha x_i^{2\omega}+1}-\sum_{i=1}^{n}x_i^{\omega}\ln(x_i)=0 \quad (12)$$

The Maximum Likelihood Estimates (MLEs), $\hat{\alpha},\ \hat{\theta}\ and\ \hat{\omega}$ of α, θ and ω are algebraic solutions of equation (10), (11) and (12) respectively. Obviously, analytical expressions for $\hat{\alpha},\ \hat{\theta}\ and\ \hat{\omega}$ are not available. Therefore, we computed the MLEs numerically using the *nloptr* package in R software (R Core Team, 2020).

5. Application

The application of PGAD is here illustrated by applying it to two datasets. Its performance was compared with that of the Generalized Akash Distribution (GAD), Power Pranav Distribution (PPD) and Topp-Leone Lomax distribution (TLLoD). We used Akaike Information Criteria (AIC), Consistent AIC (CAIC) and Bayesian Information Criteria (BIC) for the model selection criteria. Obviously, the model with the smallest value of the information criteria is the best. However, we have the -2loglikelihood values (-2logL) presented as well. The pdf of the models considered in this study are presented in Table 1.

Table 1: Some Existing Distributions

Name of distributions	**Probability density functions**	**Introducers / Authors**
Topp-Leone Lomax distribution (TLLoD)	$f(x)=2\text{abc}(1+cx)^{-(2b+1)}[1-(1+cx)^{-2b}]^{a-1}$	Oguntunde et al. (2019)
Generalized Akash Distribution (GAD)	$f_{ga}(x)=\frac{\theta^3}{2\alpha+\theta^2}(\alpha y^2+1)e^{-\theta y}$	Shanker et al. (2018)

Power Pranav Distribution (PPD)	$f(x) = \frac{\omega\theta^4 x^{\omega-1}}{(2+\theta^4)}(\theta + x^{3\omega})e^{-\theta x^{\omega}}$	Shukla (2019)
Alpha Power Transformed Power Inverse Lindley (APTPIL)	$f(x) = \begin{cases} \frac{\log(\alpha)\,\beta\theta^2}{(\alpha-1)(\theta+1)}\left(\frac{1+x^{\beta}}{x^{2\beta+1}}\right)e^{-\frac{\theta}{x^{\beta}}}\,\alpha^{\left(1+\frac{\theta}{(\theta+1)x^{\beta}}\right)e^{-\frac{\theta}{x^{\beta}}}}, if\ \alpha>0, \alpha\neq 1 \\ \frac{\beta\theta^2}{(\theta+1)}\left(\frac{1+x^{\beta}}{x^{2\beta+1}}\right)e^{-\frac{\theta}{x^{\beta}}}\ if\ \alpha = 1 \end{cases}$	Eltehiwy (2020)

Dataset 1: The data are the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 2.16, 2.2 2, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 1.46, 1.53, 1.59, 1. 6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 1.08, 1.08, 1.09, 1.12, 1.1 3, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 4.32, 4.58, 5.55.

Table 2: Comparison of the models fit to Dataset 1

Models	Parameter estimates	-2logL	AIC	CAIC	BIC
PGAD	$\hat{\alpha} = 46.8338$ $\hat{\theta} = 1.5882$ $\hat{\omega} = 1.0659$	187.6308	193.6308	193.9837	196.1841
GAD	$\hat{\alpha} = 85.032$ $\hat{\theta} = 1.6782$	188.0386	192.0386	192.2125	196.592
PPD	$\hat{\alpha} = 1.3489$ $\hat{\theta} = 1.3002$	206.4012	210.4012	210.5751	214.9545
TLLoD	$\hat{\alpha} = 3.7940$ $\hat{\theta} = 21.8742$ $\hat{\beta} = 0.0271$	188.6663	194.6663	195.0192	197.2196

The result of the analysis of the first dataset is provided in Table 2, which shows that the proposed PGAD fits the data better than the other distributions. However, it is worth to note that the AIC value of GAD (192.0386) is lower than that of PGAD (193.6308), the same cannot be said of BIC and '-*2logL*'. This because AIC penalizes model with more parameter.

Eltehiwy (2020) recently applied Alpha Power Transformed Power Inverse Lindley (APTPIL) on this same data. The author obtained the model's AIC to be "207.931" (see Eltehiwy (2020) for details), which is far higher than that of PGAD (AIC = 193.63). Hence, the PGAD fits better than this recent model.

Dataset 2: The data are service times of 63 aircraft windshield that had not failed at the time of observation. The unit for measurement is 1000h from Tahir et al. (2015). The data are:
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015,1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500,1.010, 2.141, 3.622, 1.085, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622,1.978, 3.003, 0.900, 2.053, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 3: Comparison of Distributions for Dataset 2

Models	Parameter estimates	-2logL	AIC	CAIC	BIC
PGAD	$\hat{\alpha} = 0.9485$ $\hat{\theta} = 0.8806$ $\hat{\omega} = 1.2867$	196.4145	202.4145	202.8213	204.7007
GAD	$\hat{\alpha} = 4.2202$ $\hat{\theta} = 1.2823$	199.1256	203.1256	203.3256	207.4119
PPD	$\hat{\alpha} = 1.3489$ $\hat{\theta} = 1.3002$	206.4012	210.4012	210.5751	214.9545
TLLoD	$\hat{\alpha} = 1.8943$ $\hat{\theta} = 28.1459$ $\hat{\beta} = 0.0125$	207.7056	213.7056	214.1124	215.9919

From Table 3, it has been shown that the PGAD fits the second dataset (aircraft windshield data) better than other distribution having the smallest values of the AIC, CAIC and BIC. Hence, it is clear that the proposed distribution is more flexible and provides a better fit than the other models in modeling lifetime data.

6. Conclusion

The power generalized Akash distribution is developed in this paper. Some of the basic statistical and mathematical properties of the proposed model are established, such as the moments, mean, variance, coefficient of variation, reliability, hazard, Rényi entropy and order statistics. The shape of the model could be decreasing or unimodal based on the values of α, θ and ω. The behaviour of the survival function could be decreasing, increasing or constant. The maximum likelihood estimation method was used for estimating the parameters. Two real datasets are considered to demonstrate that the new distribution can provide consistent better fit than other known competitive distributions considered. We hope that the new distribution will attract wider applications in reliability, medical sciences and other areas of research.

Acknowledgement

The authors are very grateful to the Editorial team and the two reviewers for their careful reading of the article and for their valuable suggestions that have improved the manuscript greatly.

Conflicts of Interest

The authors declare no conflict of interest.

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Appendix

R codes for computing the likelihood function, maximum likelihood estimates for the first dataset.

```
x=c(0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1.0, 1.02,
1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22,
1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68,
1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31,
2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32,
4.58, 5.55).
```

```
########################
fn<-function(parameter, x)
{
alpha=parameter[1]
theta=parameter[2]
omega=parameter[3]
L1<-log(theta^3)+log(omega)+log(x^(omega-1))
L2<-log(2*alpha+theta^2)
L3<-log(1+alpha*x^(2*omega))
L4<-(theta*x^omega)
LL=L1-L2+L3-L4
Rmle<--sum(LL)
Rmle
}
result<-optim(c(a0,b0,c0),fn,method="L-BFGS-B",lower=c(a0,b0,c0),
upper=c(Inf,Inf,Inf), x)
result
```